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A New Theory of Calculus Based on Riemann-Stieltjes Integral

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Abstract: **In this paper, we establish a new theory of calculus. Based on Riemann-Stieltjes integral, a new definition of derivative is obtained, and we study some important properties of this new calculus, such as product rule, quotient rule, chain rule, and fundamental theorem of calculus. In fact, our results are extensions of the results of ordinary calculus.**

Keywords: **New theory of calculus, Riemann-Stieltjes integral, product rule, quotient rule, chain rule, fundamental theorem of calculus.**

I. INTRODUCTION

In mathematics, the Riemann-Stieltjes integral is a generalization of the Riemann integral, named after Bernhard Riemann and Thomas Joannes Stieltjes. The integral was first defined by Stielges in 1894 [1]. It is an instructive and practical forerunner of the Lebesgue integral and a valuable tool for unifying equivalent forms of statistical theorems applicable to discrete and continuous probability.

The Riemann-Stieltjes integral appears in the original formulation of F. Riesz's theorem, which represents the dual space of the Banach space $C[a, b]$ of continuous functions in an interval $[a, b]$ as Riemann-Stieltjes integrals of functions of bounded variation. Later, that theorem was reformulated in terms of a measure. The Riemann-Stieltjes integral also appears in the formulation of the spectral theorem for (non-compact) self-adjoint (or more generally, normal) operators in a Hilbert space. In this theorem, the integral is considered with respect to a spectral family of projections [2].

Based on the Riemann-Stieltjes integral, this paper obtains a new definition of derivative, and establishes a new theory of calculus. In addition, we also studied some important properties of this new calculus, such as product rule, quotient rule, chain rule, and fundamental theorem of calculus. In fact, our results are generalizations of the results in classical calculus. The theory of Riemann-Stieltjes integral can be referred to [3-4]. For books on the theory of calculus, we can refer to [5-6].

II. PRELIMINARIES

Firstly, let's review the definition of Riemann-Stieltjes integral.

Definition 2.1: Let $f, g: [a, b] \to R$. If the limit

$$\lim_{\|\Delta\|\to 0} \sum_{k=1}^{m} f(\xi_k)\left(g(x_k) - g(x_{k-1})\right)$$

exists, where $\Delta = \{a = x_0 < x_1 < \cdots < x_m = b\}$ is a partition of the interval $[a, b]$, $\xi_k \in [x_{k-1}, x_k]$, $\Delta x_k = x_k - x_{k-1}$, and $\|\Delta\| = \max_{k=1,\cdots,m} \{\Delta x_k\}$. Then it is called the Riemann-Stieltjes integral of f with respect to g over $[a, b]$. We denote that

$$\lim_{\|\Delta\|\to 0} \sum_{k=1}^{m} f(\xi_k)\left(g(x_k) - g(x_{k-1})\right) = \int_a^b f(x) dg(x) = \int_a^b f\, dg, \tag{1}$$

and denote that $f \in R(g, [a, b])$. In particular, if $(x) = x$, then $\int_a^b f\, dg = \int_a^b f\, dx$, which is the Riemann integral of f

International Journal of Novel Research in Computer Science and Software Engineering
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on $[a, b]$.

In the following, we introduce a new definition of derivative based on Riemann-Stieltjes integral.

Definition 2.2: If $x_0 \in (a, b)$ and $f(x), g(x)$ are functions defined on (a, b). If the limit

$$\lim_{x \to x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)} \tag{2}$$

exists, then we say that f is differentiable with respect to g at x_0. If $f(x)$ are differentiable with respect to g at all $x \in (a, b)$, the we say f is a differentiable function with respect to g on (a, b), and denoted by $f \in D(g, (a, b))$. Moreover, the derivative of $f(x)$ with respect to g at x_0 is denoted by

$$f_g\ '(x_0) = \frac{d}{dg(x)} f(x)\Big|_{x=x_0} = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)}. \tag{3}$$

And we know that if $g(x) = x$, then $f_g\ '(x_0) = f'(x_0)$, which is the usual derivative of $f(x)$ at x_0. On the other hand, we define

$$f_g^{(p)}(x_0) = \frac{d^p}{dg(x)^p} f(x)\Big|_{x=x_0} = \left(\frac{d}{dg(x)}\right)\left(\frac{d}{dg(x)}\right) \cdots \left(\frac{d}{dg(x)}\right) f(x)\Big|_{x=x_0}, \tag{4}$$

the p-th order derivative of $f(x)$ with respect to g at x_0, where p is a positive integer.

Theorem 2.3 (Mean Value Theorem for Integrals): *If g is a monotone increasing function on $[a, b]$, f is a continuous function on $[a, b]$, then there is $c \in [a, b]$ such that*

$$\int_a^b f(x) dg(x) = f(c)[g(b) - g(a)]. \tag{5}$$

Proof We may assume that $g(b) \neq g(a)$. Since f is a continuous function on $[a, b]$, it follows that f have a maximum value M and a minimum value m on $[a, b]$. Since g is monotone increasing on $[a, b]$, we have

$$\int_a^b m\, dg(x) \leq \int_a^b f(x) dg(x) \leq \int_a^b M\, dg(x). \tag{6}$$

That is,

$$m\,[g(b) - g(a)] \leq \int_a^b f(x) dg(x) \leq M[g(b) - g(a)]. \tag{7}$$

Therefore,

$$m \leq \frac{\int_a^b f(x) dg(x)}{g(b)-g(a)} \leq M. \tag{8}$$

Since f is a continuous function on $[a, b]$, there is $c \in [a, b]$ such that $\frac{\int_a^b f(x) dg(x)}{g(b)-g(a)} = f(c)$, and hence the desired result holds.

III. MAIN RESULTS

In this section, we obtain some important properties of this new calculus.

Proposition 3.1: *Let λ, C be real numbers, If $f, g, h: [a, b] \to R$ and f, h are differentiable with respect to g at $x_0 \in (a, b)$, then*

$$(f + h)_g'(x_0) = f_g\ '(x_0) + h_g\ '(x_0), \tag{9}$$

$$(f - h)_g'(x_0) = f_g\ '(x_0) - h_g\ '(x_0), \tag{10}$$

$$(\lambda f)_g\ '(x_0) = \lambda f_g\ '(x_0), \tag{11}$$

$$(C)_g\ ' = 0. \tag{12}$$

Theorem 3.2: *If g is continuous at x_0, and f is differentiable with respect to g at x_0, then f is continuous at x_0.*

International Journal of Novel Research in Computer Science and Software Engineering
Vol. 12, Issue 1, pp: (37-43), Month: January - April 2025, Available at: www.noveltyjournals.com

Proof Since $\lim_{x\to x_0}[f(x)-f(x_0)] = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{g(x)-g(x_0)}\cdot \lim_{x\to x_0}[g(x)-g(x_0)] = f_g\ '(x_0)\cdot 0 = 0$, it follows that f is continuous at x_0. q.e.d.

Theorem 3.3 (Product Rule for this New Derivative): *If g is continuous at x_0, and f, h are differentiable with respect to g at x_0, then $f\cdot h$ is differentiable with respect to g at x_0 , and*

$$(f\cdot h)_g{}'(x_0) = f_g\ '(x_0)\cdot h(x_0) + f(x_0)\cdot h_g\ '(x_0). \tag{13}$$

Proof Since g is continuous at x_0 , it follows from Theorem 3.2 that h are continuous at x_0, and

$$\begin{aligned}(f\cdot h)_g{}'(x_0) &= \lim_{x\to x_0}\frac{f(x)\cdot h(x)-f(x_0)\cdot h(x_0)}{g(x)-g(x_0)}\\ &= \lim_{x\to x_0}\frac{[f(x)-f(x_0)]\cdot h(x)+f(x_0)\cdot[h(x)-h(x_0)]}{g(x)-g(x_0)}\\ &= \lim_{x\to x_0}\frac{f(x)-f(x_0)}{g(x)-g(x_0)}\cdot \lim_{x\to x_0}h(x) + f(x_0)\cdot \lim_{x\to x_0}\frac{h(x)-h(x_0)}{g(x)-g(x_0)}\\ &= f_g\ '(x_0)\cdot h(x_0) + f(x_0)\cdot h_g\ '(x_0).\end{aligned}$$

q.e.d.

Remark 3.4: It is easy to see that in Theorem 3.3, the condition ' g is continuous at x_0' can be replaced by 'function f or h is continuous at x_0'.

Theorem 3.5 (Quotient Rule): *If function h is continuous at x_0, $h(x_0)\neq 0$, and f, h are differentiable with respect to g at x_0, then $\frac{f}{h}$ differentiable with respect to g at $\frac{f(x_0)}{h(x_0)}$, and*

$$\left(\frac{f}{h}\right)_g{}'(x_0) = \frac{f_g\ '(x_0)\cdot h(x_0)-f(x_0)\cdot h_g\ '(x_0)}{h^2(x_0)}. \tag{14}$$

Proof

$$\begin{aligned}\left(\frac{f}{h}\right)_g{}'(x_0) &= \lim_{x\to x_0}\frac{\frac{f(x)}{h(x)}-\frac{f(x_0)}{h(x_0)}}{g(x)-g(x_0)}\\ &= \lim_{x\to x_0}\frac{f(x)\cdot h(x_0)-f(x_0)\cdot h(x)}{[g(x)-g(x_0)]\cdot h(x)\cdot h(x_0)}\\ &= \lim_{x\to x_0}\frac{[f(x)-f(x_0)]\cdot h(x_0)-f(x_0)\cdot[h(x)-h(x_0)]}{[g(x)-g(x_0)]\cdot h(x)\cdot h(x_0)}\\ &= \lim_{x\to x_0}\frac{1}{h(x)}\cdot \lim_{x\to x_0}\frac{[f(x)-f(x_0)]}{[g(x)-g(x_0)]} - \frac{f(x_0)}{h(x_0)}\cdot \lim_{x\to x_0}\frac{1}{h(x)}\cdot \lim_{x\to x_0}\frac{[h(x)-h(x_0)]}{[g(x)-g(x_0)]}\\ &= \frac{1}{h(x_0)}f_g\ '(x_0) - \frac{f(x_0)}{h^2(x_0)}\cdot h_g\ '(x_0)\\ &= \frac{f_g\ '(x_0)\cdot h(x_0)-f(x_0)\cdot h_g\ '(x_0)}{h^2(x_0)}.\end{aligned}$$

q.e.d.

Theorem 3.6 (Leibniz Rule): *If p is a positive integer, function g is continuous at x_0, and f, h are p times differentiable with respect to g at x_0, then*

$$(f\cdot h)_g{}^{(p)}(x_0) = \sum_{k=0}^{p}\binom{p}{k} f_g^{(k)}(x_0)\cdot h_g{}^{(p-k)}(x_0). \tag{15}$$

Where $\binom{p}{k} = \frac{p!}{k!(p-k)!}$.

Proof We use induction. The case $p = 1$ is the product rule. Assume that the case $p = m$ holds, i.e.,

$$(f\cdot h)_g{}^{(m)}(x_0) = \sum_{k=0}^{m}\binom{m}{k} f_g^{(k)}(x_0)\cdot h_g{}^{(m-k)}(x_0). \tag{16}$$

Then

$$(f\cdot h)_g{}^{(m+1)}(x_0)$$

$$= \frac{d}{dg}\left[(f\cdot h)_g{}^{(m)}\right](x_0)$$

$$= \frac{d}{dg}\left[\sum_{k=0}^{m}\binom{m}{k} f_g^{(k)}\cdot h_g{}^{(m-k)}\right](x_0)$$

$$= \left[\sum_{k=0}^{m}\binom{m}{k}\left[f_g^{(k+1)}\cdot h_g{}^{(m-k)} + f_g^{(k)}\cdot h_g{}^{(m-k+1)}\right]\right](x_0)$$

$$= \left[\sum_{k=0}^{m}\binom{m}{k} f_g^{(k+1)}\cdot h_g{}^{(m-k)} + \sum_{k=0}^{m}\binom{m}{k} f_g^{(k)}\cdot h_g{}^{(m-k+1)}\right](x_0)$$

$$= \left[f_g^{(m+1)}\cdot h_g{}^{(0)} + \sum_{k=0}^{m-1}\binom{m}{k} f_g^{(k+1)}\cdot h_g{}^{(m-k)} + f_g^{(0)}\cdot h_g{}^{(m+1)} + \sum_{k=1}^{m}\binom{m}{k} f_g^{(k)}\cdot h_g{}^{(m-k+1)}\right](x_0)$$

$$= \left[f_g^{(0)}\cdot h_g{}^{(m+1)} + \sum_{k=1}^{m}\binom{m}{k-1} f_g^{(k)}\cdot h_g{}^{(m-k+1)} + \sum_{k=1}^{m}\binom{m}{k} f_g^{(k)}\cdot h_g{}^{(m-k+1)} + f_g^{(m+1)}\cdot h_g{}^{(0)}\right](x_0)$$

$$= \left[f_g^{(0)}\cdot h_g{}^{(m+1)} + \sum_{k=1}^{m}\binom{m+1}{k} f_g^{(k)}\cdot h_g{}^{(m-k+1)} + f_g^{(m+1)}\cdot h_g{}^{(0)}\right](x_0)$$

$$= \sum_{k=0}^{m+1}\binom{m+1}{k} f_g^{(k)}(x_0)\cdot h_g{}^{(m-k+1)}(x_0).$$

Thus, the case $p = m+1$ holds. By induction, the desired result holds. q.e.d.

Theorem 3.7 (Chain Rule): *If the function h is continuous at x_0, h is differentiable with respect to g at x_0, and f is differentiable at $h(x_0)$, then the composite function $f\circ h$ is differentiable with respect to g at x_0, and*

$$(f\circ h)_g{}'(x_0) = f\ '(h(x_0))\cdot h_g{}'(x_0). \tag{17}$$

Proof : $$(f\circ h)_g{}'(x_0) = \lim_{x\to x_0}\frac{f(h(x))-f(h(x_0))}{g(x)-g(x_0)}$$

$$= \lim_{x\to x_0}\frac{f(h(x))-f(h(x_0))}{h(x)-h(x_0)}\cdot \lim_{x\to x_0}\frac{h(x)-h(x_0)}{g(x)-g(x_0)}$$

$$= \lim_{h(x)\to h(x_0)}\frac{f(h(x))-f(h(x_0))}{h(x)-h(x_0)}\cdot \lim_{x\to x_0}\frac{h(x)-h(x_0)}{g(x)-g(x_0)}\ \text{(since } h \text{ is continuous at } x_0)$$

$$= f\ '(h(x_0))\cdot h_g{}'(x_0).$$ q.e.d.

Remark 3.8: In Theorem 3.7, the condition ' h is continuous at x_0' can be replaced by 'g is continuous at x_0'.

In the following, we derive the mean value theorem for this new derivative. At first, we need some lemmas.

Theorem 3.9 (Fermat's Theorem): *Suppose that g is a strictly increasing function. If x_0 is an extreme point of f, and $f_g\ '(x_0)$ exists, then $f_g\ '(x_0) = 0$.*

Proof Since $f_g\ '(x_0)$ exists, it follows that $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{g(x)-g(x_0)}$ exists. If x_0 is a local maximum point of f, that is, $f(x_0)\geq f(x)$ on some neighbourhood of x_0. Then $\frac{f(x)-f(x_0)}{g(x)-g(x_0)}\leq 0$ if $\geq x_0$. It follows that $\lim_{x\to x_0^+}\frac{f(x)-f(x_0)}{g(x)-g(x_0)}\leq 0$. On the other hand, if $x\leq x_0$, then $\frac{f(x)-f(x_0)}{g(x)-g(x_0)}\geq 0$, and hence $\lim_{x\to x_0^-}\frac{f(x)-f(x_0)}{g(x)-g(x_0)}\geq 0$. Therefore, $\lim_{x\to x_0}\frac{f(x)-f(x_0)}{g(x)-g(x_0)} = 0$. Thus, $f_g\ '(x_0) = 0$. The case that x_0 is a local minimum point of f can be proved in the same way. q.e.d.

Theorem 3.10 (Rolle's Theorem): *Assume that g is a strictly increasing function on $[a,b]$. If f is continuous on $[a,b]$ and is differentiable with respect to g on (a,b) and $f(a) = f(b)$, then there exists $\xi\in(a,b)$ such that $f_g\ '(\xi) = 0$.*

Proof Since f is continuous on closed interval $[a,b]$, f must have a maximum value M and a minimum value m on $[a,b]$. If $M = m$, then f is a constant function, and hence $f_g\ '(\xi) = 0$ for all $\xi\in(a,b)$. If $M > m$, since $f(a) = f(b)$, it follows that there is $\xi\in(a,b)$ such that $f(\xi) = M$ or $f(\xi) = m$. And hence ξ is an extreme point of f. By Fermat's theorem, $f_g\ '(\xi) = 0$. q.e.d.

Using Rolle's theorem, we obtain the following

International Journal of Novel Research in Computer Science and Software Engineering
Vol. 12, Issue 1, pp: (37-43), Month: January - April 2025, Available at: www.noveltyjournals.com

Theorem 3.11 (Mean Value Theorem for Derivatives): *Let g be a strictly increasing function on $[a,b]$. If f is continuous on closed interval $[a,b]$ and differentiable with respect to g on open interval (a,b), then there exists $\xi \in (a,b)$ such that*

$$f(b)-f(a)=f_g{}'(\xi)[g(b)-g(a)]. \quad (18)$$

Proof Let

$$h(x)=f(x)-\left[f(a)+\frac{f(b)-f(a)}{g(b)-g(a)}[g(x)-g(a)]\right]. \quad (19)$$

Since

$$h_g{}'(x)=f_g{}'(x)-\frac{f(b)-f(a)}{g(b)-g(a)} \quad (20)$$

for all $x \in (a,b)$. Moreover,

$$h(a)=f(a)-\left[f(a)+\frac{f(b)-f(a)}{g(b)-g(a)}[g(a)-g(a)]\right]=0, \quad (21)$$

$$h(b)=f(b)-\left[f(a)+\frac{f(b)-f(a)}{g(b)-g(a)}[g(b)-g(a)]\right]=0. \quad (22)$$

It follows from Rolle's theorem that there is $\xi \in (a,b)$ such that $h_g{}'(\xi)=0$. And hence,

$$f_g{}'(\xi)=\frac{f(b)-f(a)}{g(b)-g(a)}. \quad (23)$$

Therefore,

$$f(b)-f(a)=f_g{}'(\xi)[g(b)-g(a)]. \quad \text{q.e.d.}$$

Corollary 3.12: *Suppose that g is a strictly increasing function on $[a,b]$. If f is continuous on $[a,b]$and differentiable with respect to g on open interval (a,b) such that $f_g{}'(x)=0$ for all $x \in (a,b)$. Then f is a constant function on (a,b).*

Proof If f is not a constant function on (a,b), then there exist x_1, x_2 such that

$$a<x_1<x_2<b \quad \text{and} \quad f(x_1)\neq f(x_2). \quad (24)$$

By mean value theorem for derivative, we obtain

$$f_g{}'(\xi)=\frac{f(x_2)-f(x_1)}{g(x_2)-g(x_1)} \quad (25)$$

for some $\xi \in (x_1,x_2)$. Therefore,

$$f_g{}'(\xi)\neq 0, \quad (26)$$

which is a contradiction.

Corollary 3.13: *Suppose that g is a strictly increasing function on $[a,b]$. If f,h are continuous on $[a,b]$ and differentiable with respect to g on (a,b) such that $f_g{}'(x)=h_g{}'(x)$ for all $x \in (a,b)$. Then there is a constant C such that $f(x)=h(x)+C$ for all $x \in (a,b)$.*

Proof Since $(f-h)_g{}'(x)=f_g{}'(x)-h_g{}'(x)=0$ for all $x \in (a,b)$, it follows from Corollary 3.12 that there exists a constant C such that $f(x)-h(x)=C$ *for all* $x \in (a,b)$. Therefore, the desired result holds.

Theorem 3.14: *If f,g are differentiable at x_0, and $g\ '(x_0)\neq 0$, then $f_g{}'(x)=\frac{f\ '(x_0))}{g\ '(x_0)}$.*

Proof $f_g{}'(x)=\lim\limits_{x\to x_0}\frac{f(x)-f(x_0)}{g(x)-g(x_0)}=\lim\limits_{x\to x_0}\frac{\frac{f(x)-f(x_0)}{x-x_0}}{\frac{g(x)-g(x_0)}{x-x_0}}=\frac{f\ '(x_0))}{g\ '(x_0)}$. q.e.d.

International Journal of Novel Research in Computer Science and Software Engineering
Vol. 12, Issue 1, pp: (37-43), Month: January - April 2025, Available at: www.noveltyjournals.com

Theorem 3.15 (Cauchy's Mean Value Theorem): *Assume that g is a strictly increasing function on $[a,b]$. If f,h are continuous on $[a,b]$ and differentiable with respect to g on (a,b), $h(b) \neq h(a)$, and $h_g{}'(x) \neq 0$ for all $x \in (a,b)$. Then there is $\xi \in (a,b)$ such that*

$$\frac{f(b)-f(a)}{h(b)-h(a)} = \frac{f_g{}'(\xi)}{h_g{}'(\xi)}. \tag{27}$$

Proof Let $F(x) = f(x) - \frac{f(b)-f(a)}{h(b)-h(a)} \cdot h(x)$, then

$$F_g{}'(x) = f_g{}'(x) - \frac{f(b)-f(a)}{h(b)-h(a)} \cdot h_g{}'(x)\,. \tag{28}$$

Since

$$F(a) = f(a) - \frac{f(b)-f(a)}{h(b)-h(a)} \cdot h(a) = \frac{f(a)h(b)-f(b)h(a)}{h(b)-h(a)}, \tag{29}$$

$$F(b) = f(b) - \frac{f(b)-f(a)}{h(b)-h(a)} \cdot h(b) = \frac{f(a)h(b)-f(b)h(a)}{h(b)-h(a)}. \tag{30}$$

Thus, $F(a) = F(b)$. By Rolle's theorem, there exists $\xi \in (a,b)$ such that $F_g{}'(\xi) = 0$. Therefore,

$$f_g{}'(\xi) - \frac{f(b)-f(a)}{h(b)-h(a)} \cdot h_g{}'(\xi) = 0\,. \tag{31}$$

Hence, the desired result holds. q.e.d.

Theorem 3.16 (Fundamental Theorem of Calculus): *If g is a strictly increasing function on $[a,b]$, and f is continuous on $[a,b]$, then*

(I) *$G(x) = \int_a^x f(x)dg(x)$ is differentiable with respect to g on (a,b), and*

$$G_g{}'(x) = \frac{d}{dg(x)} \int_a^x f(x)dg(x) = f(x) \tag{32}$$

for all $x \in (a,b)$.

(II) *If $F(x)$ is continuous on $[a,b]$ and differentiable with respect to g on (a,b) with $F_g{}'(x) = f(x)$ for all $x \in (a,b)$, then*

$$\int_a^b f(x)dg(x) = F(b) - F(a). \tag{33}$$

Proof (I) $G_g{}'(x^+) = \lim_{t\to x^+} \frac{G(t)-G(x)}{g(t)-g(x)}$

$$\begin{aligned}
&= \lim_{t\to x^+} \frac{\int_a^t f(u)dg(u) - \int_a^x f(u)dg(u)}{g(t)-g(x)} \\
&= \lim_{t\to x^+} \frac{\int_x^t f(u)dg(u)}{g(t)-g(x)} \\
&= \lim_{t\to x^+} \frac{f(\xi)[g(t)-g(x)]}{g(t)-g(x)} \quad \text{(where } \xi \in (x,t) \text{ by mean value theorem for integrals)} \\
&= \lim_{t\to x^+} f(\xi) \\
&= f(x).
\end{aligned} \tag{34}$$

Also, we have $G_g{}'(x^-) = f(x)$, and hence $G_g{}'(x) = f(x)$ for all $x \in (a,b)$.

(II)Since $F_g{}'(x) = G_g{}'(x)$ for all $x \in (a,b)$, it follows from Corollary 3.13 that

$$F(x) = G(x) + C \tag{35}$$

for some constant C. Since $F(a) = G(a) + C = C$, it follows that

$$F(b) = G(b) + F(a). \tag{36}$$

Therefore,

$$\int_a^b f(x)dg(x) = G(b) = F(b) - F(a). \qquad \text{q.e.d.}$$

Remark 3.17: We provide another proof of part (II) of Theorem 3.16 as follows:

If $\Delta = \{a = x_0 < x_1 < \cdots < x_m = b\}$ is any partition of the interval $[a,b]$, $\Delta x_k = x_k - x_{k-1}$ for $k = 1,\cdots,m$, and $\|\Delta\| = \max_{k=1,\cdots,m}\{\Delta x_k\}$. Then

$$F(b) - F(a) = \lim_{\|\Delta\|\to 0} \sum_{k=1}^{m} F(x_k) - F(x_{k-1})$$

$$= \lim_{\|\Delta\|\to 0} \sum_{k=1}^{m} \frac{F(x_k)-F(x_{k-1})}{g(x_k)-g(x_{k-1})} \cdot [g(x_k) - g(x_{k-1})]$$

$$= \lim_{\|\Delta\|\to 0} \sum_{k=1}^{m} F_g\,'(\xi_k) \cdot [g(x_k) - g(x_{k-1})] \quad \text{(where } \xi_k \in [x_{k-1}, x_k] \text{ by mean value theorem for derivatives)}$$

$$= \lim_{\|\Delta\|\to 0} \sum_{k=1}^{m} f(\xi_k) \cdot [g(x_k) - g(x_{k-1})] \quad \text{(since } F_g\,'(x) = f(x) \text{ for all } x \in (a,b))$$

$$= \int_a^b f(x)dg(x)\,. \qquad \text{q.e.d.}$$

IV. CONCLUSION

In this paper, we obtain a new definition of derivative based on Riemann-Stieltjes integral. Some important properties of this new calculus is studied, such as product rule, quotient rule, chain rule, and fundamental theorem of calculus. Moreover, our results are generalizations of classical calculus results. In the future, we will continue to use this new calculus to solve the problems in engineering mathematics and ordinary differential equations.

REFERENCES

[1] T. J. Stieltjes, Recherches sur les fractions continues, Ann. Fac. Sci. Toulouse, VIII: 1-122, 1894.

[2] F. Riesz, B. Sz. Nagy, Functional Analysis, Dover Publications, 1990.

[3] T. M. Apostol, Mathematical Analysis, Second Edition, Pearson Higher Education, 1974.

[4] W. Rudin, Principles of Mathematical Analysis, Third Edition, McGraw-Hill, Inc, 1976.

[5] R. Larson, B. H. Edwards, Calculus, Ninth Edition, Brooks/Cole, Cengage Learning, 2010.

[6] P. D. Lax, M. S. Terrell, Calculus With Applications, Springer-Verlag, New York, 2014.